JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 4, November 2012

SYMPLECTIC DIFFEOMORPHISMS WITH ORBITAL SHADOWING

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Abstract. We show that if a symplectic diffeomorphism has the C^1-robustly orbital shadowing property, then the diffeomorphism is Anosov.

1. Introduction

The notion of pseudo orbits often appears in several methods of the modern theory of dynamical system ([7]). Moreover, the pseudo orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory. It is well-known that if a diffeomorphism f satisfies Axiom A and the strong trasversality condition, then f has the shadowing property([7, 10]). Since such systems are structurally stable, there exists C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property because g is conjugated to f. We say that f has the C^1-robustly shadowing property if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property. Sakai proved in [11] that if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property, then f satisfies both Axiom A and the strong transversality condition. Thus the C^1-robustly shadowing property is charaterized as the set of diffeomorphisms satisfying both Axiom A and the strong

Received August 16, 2012; Accepted October 10, 2012.
2010 Mathematics Subject Classification: Primary 37C20, 37C50; Secondary 37C40, 34D05.
Key words and phrases: shadowing, weak shadowing, orbital shadowing, elliptic point, hyperbolic.
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The first author was supported by National Research Foundation of Korea (NRF) grant funded by the Korea government(No. 2011-0015193). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0007649).

transversality condition. In [9] the authors showed that if a diffeomorphism has the C^1-robustly orbital shadowing property, then it is structurally stable. It is clear that the shadowing property is the orbital shadowing property by definition, but the converse is not true. Indeed, consider a diffeomorphism f of the two-dimensional torus $\mathbb{T}^2$ studied in [8]. The nonwandering set $\Omega(f)$ consists of 4 hyperbolic fixed points, $\Omega(f) = \{p_1, p_2, p_3, p_4\}$, where p_1 is a sink, p_4 is a source, and p_2, p_3 are saddle such that $W^s(p_2) \cup \{p_3\} = W^u(p_3) \cup \{p_2\}$. It is assumed that the eigenvalues of $Df(p_2)$ are $-\mu, \nu$ with $\mu > 1, 0 < \nu < 1$, and the eigenvalues of $Df(p_3)$ are $-\lambda, \kappa$ with $\kappa > 1, 0 < \lambda < 1$. It follows from the result of [11] that f does not have the shadowing property. Plamenevskaya showed that f has the weak shadowing property if and only if the value $\log(\lambda)/\log(\mu)$ is irrational. It has seen that f has the orbital shadowing property([9]).

2. Basic definitions

Let M be a closed C^∞ $2n$-dimensional manifold with Riemannian structure and endowed with a symplectic form ω, and let $\mathrm{Diff}_\omega(M)$ be the set of symplectomorphisms, that is, of diffeomorphisms f defined on M and such that

$$\omega_x(v_1, v_2) = \omega_{f(x)}(D_x f(v_1), D_x f(v_2)),$$

for $x \in M$ and $v_1, v_2 \in T_xM$. Consider this space endowed with the C^1 Whitney topology. It is well-known that $\mathrm{Diff}_\omega(M)$ is a subset of all C^1-volume-preserving diffeomorphisms. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. By the theorem of Darboux([5, Theorem 1.8]), there is an atlas $\{\varphi_i^j : U_i \to \mathbb{R}^{2n}\}$, where U_i is an open set of M satisfying $\varphi_i^*\omega_0 = \omega$ with $\omega_0 = \sum_{i=0}^n dy_i \wedge dy_{n+i}$.

Let $f \in \mathrm{Diff}_\omega(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b(-\infty \le a < b \le \infty)$ in M is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^b(a < b)$ of f such that $x_a = x$ and $x_b = y$. Let $\Lambda \subset M$ be a closed f-invariant set. We say that f has the *shadowing property* on Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. Denote by $\mathcal{O}_f(x)$ the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ for $x \in M$. We say that f has the *weak shadowing property* on Λ (or Λ is weak shadowable for

f) if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there exists a point $y \in M$ such that $\xi \subset B_\epsilon(\mathcal{O}_f(y))$, where $B_\epsilon(A) = \{x \in M : d(x, A) < \epsilon\}$. Note that every diffeomorphism having the shadowing property has the weak shadowing property but the converse is not true. Indeed, an irrational rotation map ρ on the unit circle has the weak shadowing property but ρ does not have the shadowing property. From now, we introduce the notion of the orbital shadowing property. We say that f has the *orbital shadowing property* on Λ (or Λ is *orbital shadowable*) if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, we can find a point $y \in M$ such that

$$d_H(\overline{\mathcal{O}_f(y)}, \overline{\xi}) < \epsilon,$$

where $\overline{A}$ is the closure of a set A, and d_H is the Hausdorff distance on the set of compact subsets of M. Actually, this means that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \ \text{ and } \xi \subset B_\epsilon(\mathcal{O}_f(y)),$$

where $B_\epsilon(A)$ denotes the ϵ-neighborhood of a set $A \subset M$. We say that f has the *C^1-robustly orbitally shadowing property* if there is a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\omega(M)$ of f such that for any symplectomorphism $g \in \mathcal{U}(f)$, g has the orbital shadowing property. We denote by $\mathcal{OS}_\omega(M)$ the open subset of C^1-robustly orbitally shadowing symplectomorphisms in M. Note that f has the orbital shadowing property if and only if f^n has the orbital shadowing property, for all $n \in \mathbb{Z}$. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \ \text{ and } \ \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

Recently, Lee and Lee [4] proved that C^1-robustly orbital shadowing in volume preserving diffeomorphisms is Anosov. In this paper, a different approach must be used for volume preserving difffeomphisms. We study a symplectic diffeomorphism and orbital shadowing. Very recently, in [1], Bessa proved that if a symplectic diffeomprphism has the C^1-stably shadowing property, then the diffeomorphism is Anosov. Bessa and Vaz [2] proved that if a symplectic diffeomprphism has the C^1-stably weakly shadowing property, then M admits a partially hyperbolic splitting. In this paper, the following fact is the main result.

THEOREM 2.1. *If $f \in \mathrm{Diff}_\omega(M)$ has the C^1-robustly orbital shadowing property, then f is Anosov.*

3. Proof of Theorem 2.1

Let M be as before, and let $f \in \mathrm{Diff}_\omega(M)$. Then the following is symplectic version of Franks' Lemma.

LEMMA 3.1. [3, Lemma 5.1] *Let $f \in \mathrm{Diff}_\omega(M)$ and $\mathcal{U}(f)$ be given. Then there are $\delta_0 > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_n\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_n\}$ and symplectic maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - Dg(x_i)\| < \delta_0$ for all $1 \le i \le n$, there are $\epsilon_0 > 0$ and $\widetilde{g} \in \mathcal{U}(f)$ such that*

(a) *$\widetilde{g}(x) = g(x)$ if $x \in M \setminus U$,*
(b) *$\widetilde{g}(x) = \varphi_{g(x_i)} \circ L_i \circ \varphi_{x_i}^{-1}(x)$ if $x \in B_{\epsilon_0}(x_i)$,*

where $B_{\epsilon_0}(x_i)$ is the ϵ_0-neighborhood of x_i.

A *periodic point* for f is a point $p \in M$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the minimum period of p. We say that a periodic point is *elliptic* if $D_p f^{\pi(p)}$ has one non real eigenvalues of norm one, and if for a periodic point p of period $\pi(p)$ the tangent map $D_p f^{\pi(p)}$ has exactly $2k$ simple non-real eigenvalues of norm 1 and the other ones have norm different from 1, then we say that p is a *k-elliptic periodic point.* In dimension 2, then 1-elliptic periodic points are actually elliptic. We say that p is is *hyperbolic* if $Df^{\pi(p)}$ has no norm one eigenvalue. We say that f is in $\mathcal{F}_\omega(M)$ if there exists a neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}_\omega(M)$ such that for any $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic. To prove Theorem 2.1, we need the following Lemma.

LEMMA 3.2. [6] *If $f \in \mathcal{F}_\omega$, the f is Anosov.*

By a result of Newhouse [6] if the symplectic diffeomorphisms is not Anosov then 1-elliptic points can be created by an arbitrary small C^1-perturbations of the symplectic diffeomorphism. The following facts enough to prove Theorem 2.1 by Lemma 3.2.

LEMMA 3.3. *Let $f \in \mathcal{OS}_\omega(M)$, and $\mathcal{U}_0(f) \subset \mathrm{Diff}_\omega(M)$ be given by Lemma 3.1 with respect to $\mathcal{U}_0(f)$. Then for any $g \in \mathcal{U}(f)$, g does not have elliptic points.*

Proof. We will derive a contradiction. Suppose that there is a $g \in \mathcal{U}_0(f)$ such that g have a periodic elliptic point p. To simplify, we may assume that $g(p) = p$. Then $D_p g$ has n pairs of non-real eigenvalues, that is, $|z_i| = |\overline{z}_i| = 1$, $i = 1, \ldots, n$ with $T_pM = E_p^{L_1} \oplus \cdots \oplus E_p^{L_n}$ and $\mathrm{dim} E_p^{L_i} = 2$, $i = 1, \ldots, n$. By Lemma 3.1, there are $\alpha > 0$ and $g_1 \in \mathcal{U}(f)$

such that

$$g_1(x) = \begin{cases} \varphi_{g(p)} \circ D_p g \circ \varphi_p^{-1}(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Now, we consider the case $E_p^{L_1}(\alpha)$ other case is similar. Since p is nonhyperbolic for g_1, by our construction, we may assume that there is $l > 0$ such that $D_p g_1^l(v) = v$ for any $v \in E_p^{L_1}(\alpha) \cap \varphi_p^{-1}(B_\alpha(p))$. Take $v \in E_p^{L_1}(\alpha)$ such that $\|v\| = \alpha/4$. Then we can find a small arc $\mathcal{I}_v = \varphi_p(\{tv : 1 \le t \le 1+\alpha/4\}) \subset \varphi_p(B_\alpha(p))$ such that (i) $g_1^i(\mathcal{I}_p) \cap g_1^j(\mathcal{I}_p) = \emptyset$ if $0 \le i \ne j \le l-1$, and (ii) $g_1^l(\mathcal{I}_p) = \mathcal{I}_p$, that is, $g_1^l|_{\mathcal{I}_p}$ is the identity map. Then we can choose $0 < \epsilon < \alpha/4$ sufficiently small such that $B_\epsilon(g_1^i(\mathcal{I}_p)) \cap B_\epsilon(g_1^j(\mathcal{I}_p)) = \emptyset$ for all $1 \le i \ne j \le l-1$. Let $0 < \delta < \epsilon$ be the number of the definition of the orbital shadowing property of g_1 for ϵ. Now we construct a δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \mathcal{I}_p$ as follows; (i) we choose a finite pseudo orbit $\{v_i\}_{i=0}^k \subset \{tv : 1 \le t \le (1+\alpha/4)\}$ for some $k > 0$ such that $v_k = (1+\alpha/4)v$ and $|v_i - v_{i+1}| < \delta$ for all $0 \le i \le k-1$. Then, we get that (i) $g_1^i(\varphi_p(v)) = x_i$ for $i < 0$, (ii) $g_1^j(\varphi_p(v_i)) = x_{ml+j} = x_j$, for $0 \le m \le k-1, 0 \le j \le l-1$, and (iii) $x_i = g_1^{i-lk}(\varphi_p(v_i))$ for $k \ge lk$. Thus $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \mathcal{I}_p$ is a δ-pseudo orbit of g_1. Since g_1 has the orbital shadowing property, g_1^l has the orbital shadowing property. For simplify, we assume that $g_1^l = g_1$. Since g_1 has the orbital shadowing property, we can choose a point $y \in M$ such that

$$\xi \subset B_\epsilon(\mathcal{O}_{g_1}(y)) \text{ and } \mathcal{O}_f(y) \subset B_\epsilon(\xi).$$

Since g_1 has the orbital shadowing property, we consider two cases (i) a shadowing point $y \in \mathcal{I}_p$, and (ii) a shadowing point $y \in M \setminus \mathcal{I}_p$.

First case, let $y \in \mathcal{I}_p$. Since $g_1|_{\mathcal{I}_p}$ is the identity map, for all $n \in \mathbb{Z}$, $g_1^n(y) = y$. Then we can find $j \in \mathbb{Z}$ such that

$$d(\mathcal{O}_{g_1}(y), x_j) = d(y, x_j) > \epsilon.$$

This is a contradiction.

Finally, let $y \in M \setminus \mathcal{I}_p$ such that $y \in \varphi_p(E_p^{L_i}(\alpha)) \cap B_\alpha(p(= x_0))$ for $i = 2, \ldots, n$. Then we may assume that there are m_i(the minimum number) such that $D_p g_1^{m_i}(v) = v$ for any $v \in E_p^{L_i}(\alpha)) \cap \varphi_p^{-1}(B_\alpha(p(= x_0))), i = 2, \ldots, n$. Let $K = \mathrm{lcm}\{m_i : i = 2, \ldots, n\}$. Here lcm is the lowest common multiple. To simplify, we assume that $g_2 = g_1^K$. Then we can see that $g_2^i(y) \in B_\epsilon(x_0)$ for all $i \in \mathbb{Z}$, and since $D_p g_1^{m_i}(v) = v$ for any $v \in E_p^{L_i}(\alpha)) \cap \varphi_p^{-1}(B_\alpha(p)), i = 2, \ldots, n$, by the above argument, there is $j \in \mathbb{Z}$ such that $d(y, x_j) > \epsilon$. This is a contradiction. □

End of the proof of Theorem 2.1. Let $\mathcal{U}(f)$ be given by the definition of the C^1-robustly orbital shadowing property. Suppose that $f \notin \mathcal{F}_\omega(M)$. Then there is $g \in \mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that g have a periodic elliptic point p. By Lemma 3.3, g does not have a periodic elliptic point. This is a contradiction. Thus, if $f \in \mathcal{OS}_\omega(M)$ then $f \in \mathcal{F}_\omega(M)$. By Lemma 3.2, f is Anosov. $\square$

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